



Finding the *Threshold of Exclusion* for all single seat and multi-seat scoring rules: Illustrated by results for the Borda and Dowdall rules



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ABSTRACT

We provide a general analytic approach for calculating the *Threshold of Exclusion* (TE) for the single-seat and the multi-seat versions of all *scoring rules*, an infinitely large class of voting systems (Fishburn, 1973; Young, 1975; Saari, 1994, 1995). We offer specific results for two rules used for parliamentary elections at the national level: Borda (Black, 1958), used for national elections to special reserved seats for Hungarian and Italian ethnic minorities in Slovenia, and the unique and little known electoral system used for legislative elections on the Pacific Island state of Nauru, the *Dowdall rule*. When voters are required to provide a complete ranking of all candidates, we find that Borda does not, in general, operate as a majoritarian system in that a supermajority of roughly 2/3rd is required to guarantee electing a candidate of choice. In contrast, we find that, as district magnitude increases, the *Threshold of Exclusion* for the Dowdall rule tends to zero, in the same way as do list systems of proportional representation (PR). However, in contrast to the case for list PR rules, TE for Dowdall can still be relatively close to 1/2 for small district magnitudes.

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1. Introduction

In the political science literature on electoral systems the most common way to categorize voting rules involving competition among political parties that translate a party's candidates' share of votes into a share of seats in the legislature for the party is to locate rules along a proportionality continuum (Gallagher, 1992). This continuum is anchored by list methods of proportional representation at one end, and by plurality and majoritarian rules at the other end. Except when the voting rule is by *plurality bloc voting* or its relatives, i.e., plurality in a multi-seat constituency, expected proportionality increases with increasing *district magnitude*, the number of seats elected from any given constituency, usually denoted M ,

In political science, a standard way to locate voting rules on the proportionality continuum via a single number is to use the voting rule's *Threshold of Exclusion*. Rae et al. (1971) define the *Threshold of Exclusion* (TE) as the largest vote share a party can receive and still be denied any seats in the district (see also Rae 1967, 1971).

If TE is 1/2 we have a majoritarian system; as TE declines toward zero we move toward more perfect proportionality. If TE increases toward 1, then the voting rule approaches unanimous consent. For all common voting rules, the TE value for that rule is a function of M , and TE decreases as M increases (Lijphart and Gibberd, 1977).

The *Threshold of Exclusion* captures what is usually regarded as the single most important feature of any voting rule, but different voting rules may share the same TE value. For example, the *single transferable vote*, a procedure which requires voters to rank order candidates (see e.g., Bowler and Grofman, 2000), has the same TE value, $1/(M + 1)$, as the *D'Hondt* form of list proportional representation, in which voters cast by a single X indicating which party they wish to support.¹

¹ The D'Hondt rule for allocating seats as function of votes is mathematically identical to the *Jefferson rule* for apportioning seats within a legislature to the various units in accord with their population share. While there are a number of different ways to calculate outcomes under the D'Hondt rule perhaps the simplest is to take the shares of each of the parties (or populations) and divide it by the integers 1, 2, 3, . . . , etc. to create a column of values for each party. If there are M seats to be filled, then allocate those seats to the parties with the M highest overall quotients resulting from this procedure, with each party given one seat when one of these M highest quotients is found in its column (Balinski and Young, 1982).

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Calculating the value of TE for any voting rule involves a hypothetical worst case scenario in terms of the distribution of votes among the other parties that will minimize the given party's seat share.² The *Threshold of Exclusion* has been calculated for all of the better known voting rules used at the national parliamentary level (Rae et al., 1971; Grofman, 1975; Lijphart and Gibberd, 1977; Gallagher, 1992).

The focus of this research note is on what social choice theorists call *scoring rules*, an infinitely large class of voting systems (Fishburn, 1973; Young, 1975; Saari, 1994, 1995). A *scoring rule* operates over a set of ballots in which each voter provides a (partial or complete) ranking of a set of n alternatives. Each scoring rule can be identified with a vector of ranking weights (w_1, w_2, \dots, w_n) where w_i is the weight to be given to an alternative located at the i th rank. The only requirement is that these weights must be monotonically non-increasing.³ Let r_{jk} equal the rank given to the j th alternative by the k th voter. For any scoring rule, the cumulative score for the j th alternative is given by the weighted sum over all voters ($k = 1, n$) of $\sum w_i$ for the cases where $r_{jk} = i$. In other words, the combined score for the j th alternative is the weighted sum over all voters, k , of the weight corresponding to the rank each voter has assigned to the j th alternative. All scoring rules can be used either to select a single candidate or multiple candidates, with the M candidates with the highest scores chosen.

The two best known scoring rules are *plurality* and the *Borda count*. To represent the plurality rule as a scoring rule for the case of $M = 1$, we use the normalized weights vector $(1, 0, 0, \dots, 0)$, i.e., only first preferences are given any weight.⁴ While usually thought of as a procedure to make choices within a small committee, the Borda count⁵ has been used for national elections to special reserved legislative seats for Hungarian and Italian ethnic minorities in Slovenia (Toplak, 2006). If we are choosing among n alternatives, one way to think of the Borda count is to imagine that each voter awards a given alternative one vote for each alternative the voter prefers to that alternative, and we then sum the counts for each alternative across the set of voters (Young, 1974). A standard way of representing the Borda rule as a scoring rule is as the vector of weights $(n - 1, n - 2, \dots, 2, 1, 0)$. In this representation it is clear that the Borda rule requires the voters to assign a rank to each alternative.

Calculating the TE value for plurality is trivial, as is the case for any other voting rule that does not require rankings⁶; however, the

TE formula for the Borda rule has not previously been calculated, nor have TE values been calculated for other scoring rules that involve rankings. Our goal is to use the *Threshold of Exclusion* as a way to categorize all possible scoring rules with respect to the proportionality continuum to see how close their TE values come to the limiting cases of $TE = 0$ and $TE = 0.5$. After first presenting a general four component formula that allows us to calculate TE values for any scoring rule, we will illustrate results for plurality, the Borda rule, and for another scoring rule involving ranked preferences that has had real world use in parliamentary elections, the *Dowdall system* used in the Pacific Island state of Nauru. With the Dowdall rule, a first preference is worth 1, a second preference 1/2, a third preference 1/3, a fourth preference 1/4 and so on.⁷ In other words, to represent the Dowdall rule as a scoring rule, we have a weighting vector of $(1, 1/2, 1/3, \dots, 1/n)$.⁸

2. Finding a general formula for the threshold of exclusion of scoring rules

While TE has been calculated for methods using ranked voting, such as the single transferable vote (Lijphart and Gibberd, 1977), there are complexities when voters must rank preferences that need to be taken into account. Nonetheless, we are still dealing with a zero sum game in which we may partition the political parties into two sets: one seeking to elect its most preferred candidate, the other seeking to prevent that from happening. Although the Borda rule and the Dowdall rule involve ranked ballots and plurality does not, we are able to specify a four component model which applies to all three, and to any other possible scoring rule specified in vector terms. In the discussion that follows we will use the term 'score' as synonymous with 'weight,' and we assign n to be the total number of voters, and M to be district magnitude.

The four components are:

- (1) the score given to a first place ranking
- (2) the score given to a last place ranking
- (3) the average of the last $n - 1$ scores
- (4) the average of the first M scores.

Theorem. *If voters must cast a ballot containing a full ranking of all candidates,⁹ for any scoring rule, its Threshold of Exclusion, TE, is the minimum x such that the inequality below is satisfied, where x is the vote share of one of the blocs and $1 - x$ is the vote share of the opposing bloc. Note that, for most scoring rules, TE depends upon x .*

$$\begin{aligned}
 &x * (\text{first place score}) + (1 - x) * (\text{last place score}) \\
 &> x * (\text{average of last } n - 1 \text{ scores}) \\
 &+ (1 - x) * (\text{average of first } M \text{ scores}). \quad (1)
 \end{aligned}$$

² Paralleling the *Threshold of Exclusion* there is also a *Threshold of Representation*, which is the smallest vote share a party can receive and still win at least one seat in the district (see Rae et al., 1971; Rae, 1971). The *Threshold of Representation* involves a hypothetical *best case* scenario in terms of the distribution of votes among the other parties that will maximize the given party's seat share.

³ Two vectors of weight are equivalent if one can be derived from the other by a simple linear transformation.

⁴ *Plurality bloc voting* is the term used in the United States to refer to plurality within multi-seat constituencies. If there are M seats to be filled then each voter has M ballots, so that we would have vector whose first M entries were 1's and the remainder of whose entries were zero.

⁵ The *Borda count* was named after the French mathematician Jean-Charles de Borda (Borda, 1781; Black, 1958; Young, 1974; Saari, 1994, 1995) who proposed its use in the 19th century for elections to the French Royal Academy.

⁶ In particular, we may treat these voting games as *zero sum games* (Owen, 1975) in which the parties who are seeking to deny the other party any seats are best coordinating the votes of their supporters behind a single candidate or slate of candidates, thus creating the equivalent of a two player game. For example, to find the *Threshold of Exclusion* for the *D'Hondt* form of list PR in which we allocate seats to the parties with the M highest quotients when we divide their vote shares by the integers 1, 2, 3, ..., M , we recognize that, in an M seat district, for a party with a share x of the vote ($0 < x < 1$) to win at least one seat, the bloc of parties with a vote share of $1 - x$ must not be large enough to win all M seats. But for *D'Hondt*, since the *D'Hondt* divisors are 1, 2, 3, ..., M , to win M seats with a vote share of $1 - x$, it is necessary that $(1 - x)/M > x$. After straightforward algebra this simplifies to $x > 1/(M + 1)$.

⁷ In its original 1971 formulation, the Dowdall system used the fractions 1, 1/2, 1/3, etc., but this was subsequently converted to a decimal basis to a fixed number of decimals. For mathematical tractability in deriving exact analytic results we will use the fractional representation.

⁸ Electoral results in Nauru under the Dowdall rule, with simulated comparisons to other rules, are presented in Fraenkel and Grofman (2014). The only other known real world example involving use of a similar inverse to rank scoring rule at the national level is for the selection of presidential nominees by the parliament of another small island nation in the Pacific, Kiribati (Trease, 2005; Reilly, 2001; Reilly, 2002).

⁹ Scoring rules such as Borda and the Dowdall rule can be implemented either by allowing all voters to rank as many candidates as they choose, or by requiring voters to rank order only a fixed number of candidates (i.e., cast what we might call a truncated ballot), or by requiring voters to rank order all candidates. In Nauru, every voter is required to provide a full preference ranking on pain of their vote being declared invalid. Here we look only at the case where voters must rank order all alternatives. By requiring a full preference ranking to be used, we avoid dealing with the complication of strategic choices about how many candidates to rank (cf. Glazer et al., 1984).

Proof. Without loss of generality, since this is the worst case scenario for the party seeking to elect its most preferred candidate, we assume that all the other parties are in opposition to this party and organized into a single bloc able to coordinate the ballots cast by all the voters supporting any party in the bloc.¹⁰ For any scoring rule, in the worst case scenario for the bloc seeking to elect its top ranked candidate, candidate A, the adversarial bloc will, if it strategizes rationally, do two things. (1) It will place the opposition's most preferred candidate last, and (2) it will use a strategy that randomizes ranks for its top M choices in the top M places to guard against the one of these who receives the fewest votes being defeated by the first bloc's first choice candidate. The best the bloc seeking to elect its top ranked candidate can do in this worst case scenario is to (1) assign its most preferred candidate, A, all of its first place votes, and (2) randomize ranks among the remaining alternatives. This randomization makes it harder for the second bloc to defeat its opponent's first choice, candidate A, by supporting the opposing bloc's 2nd through $(M + 1)$ th place alternatives so as to elect that set of M alternatives and shut out candidate A. Eq. (1) merely states this results in terms of an inequality. □

We show in Table 1 results for each of the four components for plurality, Borda and Dowdall respectively, and then the TE formula that results from combining these components as shown in Eq. (1). The *Threshold of Exclusion* for plurality is the same as for plurality bloc voting. For the Dowdall rule, TE does not have a closed form solution, and so we report results for TE for the Dowdall rule separately in Table 2, for a range of values of n and M .

Looking at the four components for plurality shown in Table 1, we get a TE value of $1/2$ for the Plurality/Plurality bloc voting case, since the algebra of Eq. (1) simplifies to $x > 1 - x$, i.e., $TE > 1/2$. For Borda, we solve the algebra for $x * (n - 1) + (1 - x) * 0 > x * (n - 2)/2 + (1 - x) * (2n - M - 1)/2$ to obtain the result for the Borda TE stated in Table 1. Examination of this formula makes it clear that is bounded from below by $1/2$ and from above by $2/3$.

To see how this works for Borda, consider the case of $M = 2$ and three alternatives {A, B, C}. We can think of there being two blocs. One bloc, of size x , wishes to elect A; the other bloc, of size $1 - x$, wishes to prevent the election of A. Each controls its own voters and so the strategically cast ballots are as shown below.¹¹

$x/2$	$x/2$	$(1 - x)/2$	$(1 - x)/2$
A	A	B	C
B	C	C	B
C	B	A	A

The *Borda score* for A is $2 * 2 * x/2 = 2x$. The Borda score for B is $x/2 + 2 * (1 - x)/2 + (1 - x)/2 = 1/2 + (1 - x)$, as is the Borda score for C. A will be excluded only if both B and C win, which occurs if $3/2 - x > 2x$, so that $1/2 > x$. More generally, if the voting vector (w_1, w_2, w_3) is used, then A receives $w_1 * x + w_3 * (1 - x)$ points, B receives $w_2 * x/2 + w_3 * x/2 + w_1 * (1 - x)/2 + w_2 * (1 - x)/2$ and C receives the same number of points as B. We may solve the resulting algebra to determine how large a vote share the first party needs to assure it elects its most preferred candidate. Doing the algebra for $M = 1, n = 3$, we find that the *Borda score* for A is $2 * 2 * x/2 = 2x$, while the Borda score for B is $x/2 + 2 * (1 - x)$.

¹⁰ In a strict party list PR system, if only a single candidate from a list is elected, the candidate who will be elected is the top of the party's list. Looking to the first preference of a bloc is the most similar equivalent of the more usual TE calculations for non-ranked procedures.

¹¹ We are indebted to an anonymous reviewer for suggesting to present results in this tabular fashion so as to make clearer the link to more familiar game theoretic/social choice ideas.

Thus $7x = 4$, or $x < 4/7 = 0.571$, the value shown in Table 2 for the size of the bloc needed to guarantee the election of its first place choice, A for this case. If $M = 2$ it might appear that the opposing bloc could just concentrate its votes on a single candidate, say B, by casting all $1 - x$ votes on the preference ranking BCA. But the bloc supporting A is guarding against this by randomizing between ABC and ACB.

Results are a bit more complicated for the Dowdall rule. Table 2 shows results for what the Dowdall *Threshold of Exclusion* would look like for the kinds of parameter values we actually see in Nauru elections ($M = 2, M = 3$ and $M = 4$; a range in numbers of candidates in recent multi-seat parliamentary elections from 5 to 22, with a mean of roughly 9), and it provides, for comparison purposes, values for single seat elections ($M = 1$) as well. Also, to make easy comparisons possible with the general formula for the Borda rule, we show the corresponding Borda TE values.

It should be apparent from Table 2 that, for $M = 1$, the TE for the Dowdall rule hardly varies at all; it reaches its maximum of about 0.541 for about $n = 5$, and declines toward 0.50 from there. For $M = 2$, the TE value for the Dowdall rule does not vary widely for different values of n . It increases slightly to a maximum of about 0.454 for $n = 8$, and decreases from there to a limit that we can determine to be $3/7$ (0.429), which is the limiting *Threshold of Exclusion* for the Dowdall rule for the $M = 2, n \gg M$ case.¹²

Table 2 also allows us to see that the rule used on Nauru requires a much smaller bloc to guarantee representation for $M = 2$ than is the case for Borda (where the TE goes from 0.50 up to a limit of $2/3$ for large n). Indeed, for the values of M used on Nauru, the Dowdall rule looks in between the Borda rule and a PR rule such as D'Hondt, which, for $M = 2$, has a *Threshold of Exclusion* of $1/3$.

For $M = 4$ it is again apparent that the TE for the Dowdall rule does not vary much with n . It starts at about 0.32 for $n = 5$ and increases slightly. It appears to be approaching its peak at about 0.353 as n gets to 15, and it then decreases toward a limit of what we can show to be $25/73 = 0.342$.¹³

In general, for $M > 1, n \gg M$, as M increases, the Dowdall *Threshold of Exclusion* goes to zero. In particular, if S is the average weight for the first M preferences, then the TE goes to $S/(1 + S)$ as n gets large. For example, for $M = 1, S$ is 1 and the TE goes to $1/2$. For $M = 2, S = 3/4$ and the TE goes to $3/7$. For $M = 4, S = 25/48$, and TE goes to $25/73$. As M gets large, S goes to zero and the limit of the TE goes to 0 for large n .

3. Conclusions

Calculating the *Threshold of Exclusion* for the general class of scoring rules, which includes ranked voting methods such as Borda and the Dowdall rule, is, as we have seen, more complicated than calculating TE for, say, list PR rules because (a) when voters are required to cast a full ballot, we must allow for the possibility that a bloc seeking to elect its most preferred candidate is casting some of its ballots for candidates favored by its opposition, (b) we must look at randomization used for strategic optimality, and (c) for some rules, such as the Dowdall rule used on Nauru, closed form solutions are not possible.¹⁴ But these problems can all readily be surmounted.

¹² The Dowdall TE formula for $M = 2$ can be rewritten as $x > (3n - 4)/(7n - 4 - 4nQ_n)$, where Q_n is the mean of the last $n - 1$ Dowdall scores. For $n = 3$, for example, Q_3 is $5/12$. However Q_n quickly tends to zero as n increases, so that the limit result is $3/7$.

¹³ The Dowdall TE formula for $M = 4$ can be rewritten as $x > (25n - 48)/(73n - 48 - 48nQ_n)$, where Q_n is, as in the previous footnote, the mean of the last $n - 1$ Dowdall scores.

¹⁴ Of course, for all voting rules, if we compare districts with different values of M , we need to remember that these vary in their population (Grofman, 2001).

Table 1

Four components used to determine the *Threshold of Exclusion* for Plurality, the Borda Count, and the Dowdall rule used on Nauru.

	Plurality	Borda	Dowdall
First place score	1	$n - 1$	1
Last place score = 0	0	0	$1/n$
Average of first M scores	1	$((n - 1) + (n - M))/2 = (2n - M - 1)/2$	$\sum_{j=1}^{j=M} (1/j)/(M)$
Average of last $n - 1$ scores	0	$(n - 2)/2$	$\sum_{j=n}^{j=2} (1/j)/(n - 1)$
TE	1/2	$(2n - M - 1)/(3n - M - 1)$	See Table 2

Table 2

Thresholds of Exclusion for the Dowdall rule and for Borda: ($M = 1, 2, 4$ and various values of n).

n	Dowdall TE, $M = 1$	Dowdall TE, $M = 2$	Dowdall TE, $M = 4$	Borda TE, $M = 1$	Borda TE, $M = 2$	Borda TE, $M = 4$
2	0.500			0.500		
3	0.533	0.417		0.571	0.500	
4	0.540	0.439		0.600	0.556	
5	0.541	0.447	0.321	0.615	0.583	0.500
6	0.541	0.452	0.333	0.625	0.600	0.538
7	0.539	0.453	0.340	0.632	0.611	0.563
8	0.538	0.454	0.345	0.636	0.619	0.579
9	0.536	0.453	0.347	0.640	0.625	0.591
10	0.534	0.453	0.349	0.643	0.630	0.600
11	0.533	0.453	0.350	0.645	0.633	0.607
12	0.532	0.452	0.351	0.647	0.636	0.613
13	0.530	0.452	0.352	0.649	0.639	0.618
14	0.529	0.451	0.352	0.650	0.641	0.622
15	0.528	0.450	0.353	0.651	0.643	0.625

(1) We have been able to state a general four component formula that applies to all scoring rules where voters must cast a full ballot ranking all candidates.

$$x * (\text{first place score}) + (1 - x) * (\text{last place score}) > x * (\text{average of last } n - 1 \text{ scores}) + (1 - x) * (\text{average of first } M \text{ scores}).$$

(2) We have been able to provide a complete closed form characterization of the *Threshold of Exclusion* for the Borda rule, namely as

$$(2n - M - 1)/(3n - M - 1).$$

(3) We have provided a general formula for calculating the TE for the Dowdall rule. While this formula is not reducible to closed form expression of the type we found for Borda, except for special cases,¹⁵ it still allows us to get exact values by doing some straightforward calculations of the subcomponents of this formula for specific values of M and n . For the Dowdall rule, for values of $M = 1, 2,$ and $4,$ we have provided values for the *Threshold of Exclusion* for up to 15 candidates.

Our results have some very interesting implications for understanding the properties of the Borda rule and the Dowdall rule in the context of legislative elections, and for appreciating the differences between the two rules. While the social choice literature is replete with sophisticated discussions of differences among forms of scoring rules (see e.g., Saari, 1994, 1995), the political science literature has almost entirely ignored this large class of scoring rules, except of course, for plurality, because they are not in use in national parliamentary elections in major democracies. In the handful of cases where there are references in the political science literature to scoring rules that requires ranking of ballots, as in the Dowdall rule, such rules are treated as if they were merely a trivial variant of the Borda rule (Reilly, 2001, 2002).

In comparing the Borda rule and the Dowdall rule, we find that the Borda rule is a *supermajoritarian* rule in generally requiring

more than a simple majority to guarantee that a bloc can elect a preferred representative. In other words, when voting a full ranking is required to cast a valid ballot, in an election where slates are run by parties, to guarantee an election under Borda of its first place candidate generates an incentive for parties to offer consensus candidates. Indeed, for $n \gg M,$ the TE for Borda goes to 2/3. This is in marked contrast to the behavior of TE for other common voting rules, where TE decreases as M increases (Lijphart and Gibberd, 1977).

In contrast, for large $M, n \gg M,$ the Dowdall *Threshold of Exclusion* approaches 0, which is the same limit as the *Threshold of Exclusion* for proportional representation rules such as the D'Hondt form of list PR. In other words, in principle, the Dowdall rule encourages a more fragmented political competition. Thus, it is an error to think of the Dowdall rule as a variant of the Borda method. While it is true that both are scoring rules; one might also say that Dowdall is a variant of plurality, since plurality, too, is a scoring rule. In fact, plurality, Borda and Dowdall are each very different as regards their *Threshold of Exclusion* values, and thus should be regarded as very different rules.¹⁶

In social choice theory, manipulability has been studied in the context of choice of voting rules, the consequences of the sequencing of alternatives in rules that proceed in some sequential fashion to eliminate alternatives, the grouping of alternatives for consideration, the number of alternatives to offer and the decision to include or exclude particular candidates, and the partitioning of voters into distinct constituencies, but with the decisions of voters to vote for candidates other than their “sincere” choices the most frequently studied form of manipulation. Here we have shown that a randomization strategy may be an important strategic tool. Both the set of voters who seek to elect their most preferred candidate A

¹⁵ See footnote 11 and 12 above.

¹⁶ Fraenkel and Grofman (2014) examine data from actual parliamentary elections in Nauru over several decades using the Dowdall rule, and data from actual elections in Slovenia using the Borda rule to elect ethnic minority representatives to the national parliament, to compare these two rules as they operate in real world settings. In practice differences between them are much slighter than the comparisons in Table 2 might suggest because voters are allowed to cast truncated ballots in Slovenia and usually did so.

and the set of voters who seek to block the election of A are shown to be voting in a straightforward fashion with respect to A; the first group puts A first on all ballots; the latter group puts A last on all ballots. But, with regard to intermediate preferences, as long as the goal of either electing A or stopping A is regarded as paramount, the randomization needed for strategic optimality trumps actual preference over those intermediate alternatives.¹⁷

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¹⁷ Thus, using the *Threshold of Exclusion* as a lens to study the properties of scoring rules allows us new insights into the long-standing debate over the merits of the Borda rule in terms of propensities for manipulability. See e.g., Apestequia et al. (2011), Dummett (1998), Emerson (2013), Saari (1990) and Tabarrok and Spector (1999).